Ensemble Control as a Tool for Robot Motion Planning:

Uncertainty, Optimality, and Complexity

Timothy Bretl
(w/ Aaron Becker)

University of Illinois at Urbana-Champaign

IEEE ICRA Workshop on Uncertainty in Automation
May 9, 2011

Outline

1. Introduction
   - Problem
   - Message
2. Linear example
3. Nonlinear example
4. Conclusion

Motion planning

Find

$$u\colon [0, t_f] \to \mathcal{U} \subset \mathbb{R}^m$$
$$x\colon [0, t_f] \to \mathcal{X} \subset \mathbb{R}^n$$

satisfying

$$\dot{x}(t) = f\,(t, x(t), u(t))$$
$$x(0) = x_{\text{start}}$$
$$x(t_f) = x_{\text{goal}}$$

for free final time t_f

Motion planning under bounded uncertainty

Find

$$u \colon [0, t_f] \to \mathcal{U} \subset \mathbb{R}^m$$
$$x \colon [0, t_f] \to \mathcal{X} \subset \mathbb{R}^n$$

satisfying

$$\dot{x}(t) = f\left(t, x(t), u(t), \epsilon\right)$$
$$x(0) = x_{\text{start}}$$
$$x(t_f) = x_{\text{goal}}$$

for free final time t_f, despite bounded uncertainty

$$\epsilon \in [1 - \delta, 1 + \delta]$$

Motion planning as ensemble control

Find

$$u\colon [0, t_f] \to \mathcal{U} \subset \mathbb{R}^m$$
$$x\colon [0, t_f] \times [1-\delta, 1+\delta] \to \mathcal{X} \subset \mathbb{R}^n$$

satisfying

$$\dot{x}(t, \epsilon) = f\left(t, x(t, \epsilon), u(t), \epsilon\right)$$
$$x(0, \epsilon) = x_{\text{start}}$$
$$x(t_f, \epsilon) = x_{\text{goal}}$$

for free final time t_f and for all

$$\epsilon \in [1-\delta, 1+\delta]$$

Take-away message

Ensemble control theory is a useful way to deal with bounded uncertainty in dynamical systems.

To steer one system with an uncertain parameter, we pretend to steer a continuous ensemble of systems, each with a particular value of that parameter.

In the examples we will consider, this approach costs us nothing in terms of computational complexity.

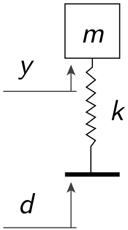
Outline

A driven harmonic oscillator

- This system is linear and has the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\epsilon & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \epsilon \end{bmatrix} u$$
$$= Ax + Bu,$$

where $(x_1, x_2) = (y, \dot{y})$, $u = d$, and $\epsilon = k/m$.



- For known ϵ, this system is controllable because

$$\begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

is full rank.

Ensemble controllability

- For unknown ϵ, consider the lifted system

$$\begin{aligned}\dot{x}(t,\epsilon) &= \begin{bmatrix} 0 & 1 \\ -\epsilon & 0 \end{bmatrix} x(t,\epsilon) + \begin{bmatrix} 0 \\ \epsilon \end{bmatrix} u(t) \\ &= A(\epsilon)x(t,\epsilon) + B(\epsilon)u(t).\end{aligned}$$

- For any integer $k \geq 0$, we have

$$\begin{bmatrix} A^{2k+1}B & A^{2k}B \end{bmatrix} = \begin{bmatrix} \epsilon^{k+1} & 0 \\ 0 & \epsilon^{k+1} \end{bmatrix}.$$

- So, we can approximate any desired movement direction by a polynomial in ϵ, with error vanishing in k:

$$f(\epsilon) \approx \sum_{i=0}^{k-1} \epsilon^i \left(a_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Control by piecewise-constant inputs (1/2)

- A discrete-time model is

$$\begin{aligned} x_d(i+1,\epsilon) &= e^{A(\epsilon)T} x_d(i,\epsilon) + \left(\int_0^T e^{As} B ds \right) u_d(i) \\ &= A_d(\epsilon) x_d(i,\epsilon) + B_d(\epsilon) u_d(i) \end{aligned}$$

- If $x_d(0,\epsilon) = 0$ then

$$x_d(2k,\epsilon) = \sum_{i=0}^{2k} A_d^i(\epsilon) B_d(\epsilon) u_d(2k-i)$$

- This can be approximated by the series expansion

$$x_d(2k,\epsilon) \approx \sum_{i=0}^{k-1} \frac{1}{i!} \left(\left. \frac{\partial^i x_d(2k,\epsilon)}{\partial \epsilon^i} \right|_{\epsilon=1} \right) (\epsilon - 1)^i$$

Control by piecewise-constant inputs (2/2)

- The result has the form

$$x_d(2k, \epsilon) = \sum_{i=1}^{k} \begin{bmatrix} r_i \\ s_i \end{bmatrix} (\epsilon - 1)^{i-1} + O\left(|\epsilon - 1|^k\right)$$

  where $r, s \in \mathbb{R}^k$ are linear in $u_d \in \mathbb{R}^{2k}$

- To achieve (x_1, x_2) with error of order k in $|\epsilon - 1|$:

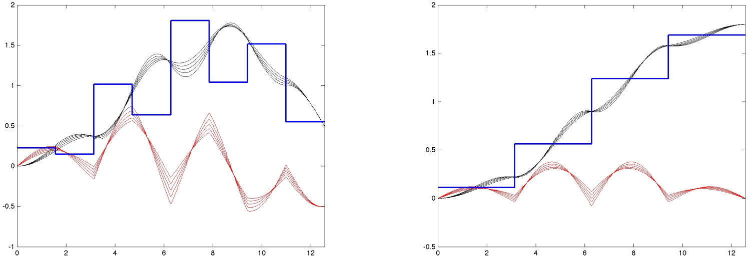
$$r = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad s = \begin{bmatrix} x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- The solution ($2k$ linear equations in $2k$ variables) has the form

$$u_d = K_1 x_1 + K_2 x_2$$

  for matrices K_1 and K_2 that can be precomputed

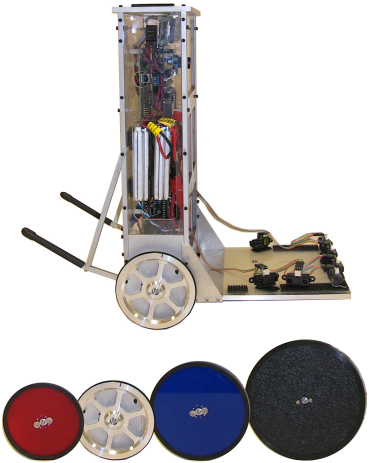
Example results in simulation



Outline

1. Introduction
2. Linear example
3. Nonlinear example
   - Model
   - Analysis
   - Results
4. Conclusion

A differential drive robot with uncertain wheel radius



$\epsilon = 0.8$ $\epsilon = 0.83$ $\epsilon = 1.0$ $\epsilon = 1.2$

For a fixed wheel separation, inputs scale with wheel radius

$\epsilon r =$ wheel radius

$b =$ wheel separation

$$v = \epsilon \left(\frac{r(\omega_R + \omega_L)}{2} \right) = \epsilon u_1$$

$$w = \epsilon \left(\frac{r(\omega_R - \omega_L)}{b} \right) = \epsilon u_2$$

Scratch-drive microrobots with uncertain forward speed

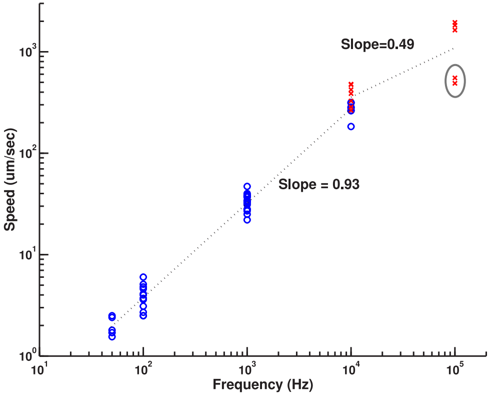


Figure: Donald et al. (2008)

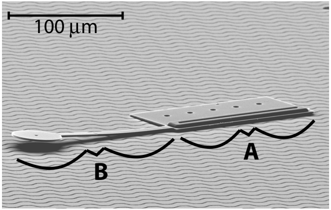


Figure: Donald et al. (2006)

- For a fixed turning radius, inputs scale with forward speed

Analysis of one robot

- This system is nonlinear and has the form

$$\dot{x} = \epsilon \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \epsilon \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$
$$= \epsilon g_1(x) u_1 + \epsilon g_2(x) u_2$$

- For known $\epsilon > 0$, this system is controllable because

$$\big[[\epsilon g_1, \epsilon g_2] \quad \epsilon g_2 \quad \epsilon g_1 \big] = \begin{bmatrix} \epsilon^2 \sin x_3 & 0 & \epsilon \cos x_3 \\ -\epsilon^2 \cos x_3 & 0 & \epsilon \sin x_3 \\ 0 & \epsilon & 0 \end{bmatrix}$$

is full rank everywhere

Analysis of an ensemble (1/3)

- For unknown ϵ, consider the lifted system

$$\begin{aligned}\dot{x}(t,\epsilon) &= \epsilon \begin{bmatrix}\cos x_3(t,\epsilon)\\ \sin x_3(t,\epsilon)\\ 0\end{bmatrix} u_1(t) + \epsilon \begin{bmatrix}0\\0\\1\end{bmatrix} u_2(t)\\ &= \epsilon g_1(x(t,\epsilon))u_1(t) + \epsilon g_2(x(t,\epsilon))u_2(t)\end{aligned}$$

- Heading is not controllable, since for all ϵ

$$x_3(t,\epsilon) = x_3(0,\epsilon) + \epsilon\theta(t) \qquad \text{where} \qquad \dot{\theta}(t) = u_2(t)$$

- Eliminate heading to get a controllable subsystem:

$$\begin{bmatrix}\dot{x}_1(t,\epsilon)\\ \dot{x}_2(t,\epsilon)\\ \dot{\theta}(t)\end{bmatrix} = \epsilon \begin{bmatrix}\cos\left(x_3(0,\epsilon)+\epsilon\theta(t)\right)\\ \sin\left(x_3(0,\epsilon)+\epsilon\theta(t)\right)\\ 0\end{bmatrix} u_1(t) + \begin{bmatrix}0\\0\\1\end{bmatrix} u_2(t)$$

Analysis of an ensemble (2/3)

- Take Lie brackets to find new control vector fields:

$$
\begin{aligned}
[\epsilon g_1, g_2] &= \epsilon \left(\frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 \right) \\
&= -\epsilon^2 \begin{bmatrix} -\sin\left(x_3(0,\epsilon) + \epsilon\theta(t)\right) \\ \cos\left(x_3(0,\epsilon) + \epsilon\theta(t)\right) \\ 0 \end{bmatrix} \\
&= -\epsilon^2 g_3 \\
\left[[\epsilon g_1, g_2], g_2\right] &= -\epsilon^3 g_1 \\
\left[\left[[\epsilon g_1, g_2], g_2\right], g_2\right] &= -\epsilon^4 g_3 \\
&\vdots
\end{aligned}
$$

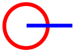
Analysis of an ensemble (3/3)

- We can approximate any desired movement direction by a polynomial in ϵ, with error vanishing in k:

$$f(\epsilon) \approx cg_2 + \sum_{i=0}^{k} \left(a_i \epsilon^{2i+1} g_1 + b_i \epsilon^{2i+2} g_3 \right)$$

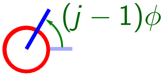
Control with piecewise-constant inputs

u_1	u_2	Δt

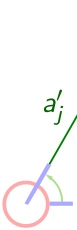


Control with piecewise-constant inputs

u_1	u_2	Δt
0	$1/\lambda$	$\lambda(j-1)\phi$

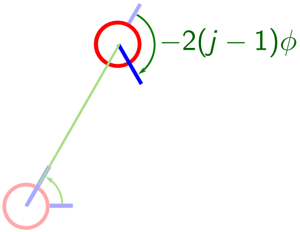


Control with piecewise-constant inputs



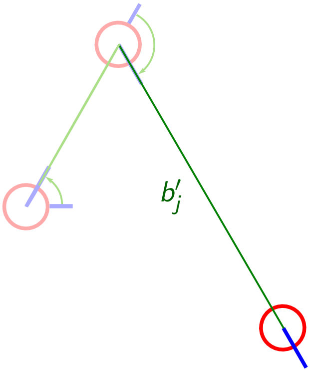
u_1	u_2	Δt
0	$1/\lambda$	$\lambda(j-1)\phi$
$\text{sign}\left(a_j'\right)$	0	$\left\lvert a_j' \right\rvert$

Control with piecewise-constant inputs



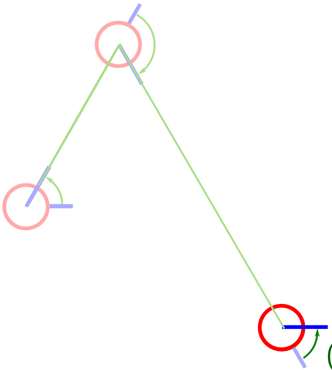
u_1	u_2	Δt
0	$1/\lambda$	$\lambda(j-1)\phi$
$\operatorname{sign}\left(a'_j\right)$	0	$\left\lvert a'_j\right\rvert$
0	$-1/\lambda$	$2\lambda(j-1)\phi$

Control with piecewise-constant inputs



u_1	u_2	Δt
0	$1/\lambda$	$\lambda(j-1)\phi$
$\text{sign}\left(a'_j\right)$	0	$\left\lvert a'_j\right\rvert$
0	$-1/\lambda$	$2\lambda(j-1)\phi$
$\text{sign}\left(b'_j\right)$	0	$\left\lvert b'_j\right\rvert$

Control with piecewise-constant inputs



u_1	u_2	Δt
0	$1/\lambda$	$\lambda(j-1)\phi$
$\text{sign}\left(a'_j\right)$	0	$\left\|a'_j\right\|$
0	$-1/\lambda$	$2\lambda(j-1)\phi$
$\text{sign}\left(b'_j\right)$	0	$\left\|b'_j\right\|$
0	$1/\lambda$	$\lambda(j-1)\phi$

Correspondence with polynomial approximation

- The result is to achieve

$$\Delta x_1(\epsilon) = (a_j' + b_j')\epsilon \cos(\epsilon(j-1)\phi)$$
$$\Delta x_2(\epsilon) = (a_j' - b_j')\epsilon \sin(\epsilon(j-1)\phi)$$
$$\Delta\theta = 0$$

- With the input transformation

$$a_j' = \frac{a_j + b_{j-1}}{2}$$
$$b_j' = \frac{a_j - b_{j-1}}{2}$$

for freely chosen $a_k, b_k \in \mathbb{R}$, we can write

$$\Delta x_1(\epsilon) = a_j \epsilon \cos(\epsilon(j-1)\phi)$$
$$\Delta x_2(\epsilon) = b_{j-1} \epsilon \sin(\epsilon(j-1)\phi)\,.$$

Sequence of motion primitives (1/3)

- For $a_{k+1} = 0$, the result after $k+1$ primitives is

$$\Delta x_1(\epsilon) = \sum_{j=1}^{k} a_j \epsilon \cos\left(\epsilon(j-1)\phi\right)$$

$$\Delta x_2(\epsilon) = \sum_{j=1}^{k} b_j \epsilon \sin\left(\epsilon j \phi\right)$$

- This can be approximated by the series expansions

$$\Delta x_1(\epsilon) \approx \sum_{i=0}^{k-1} \frac{1}{i!} \left(\left.\frac{\partial^i \Delta x_1}{\partial \epsilon^i}\right|_{\epsilon=1} \right) (\epsilon - 1)^i$$

$$\Delta x_2(\epsilon) \approx \sum_{i=0}^{k-1} \frac{1}{i!} \left(\left.\frac{\partial^i \Delta x_2}{\partial \epsilon^i}\right|_{\epsilon=1} \right) (\epsilon - 1)^i$$

Sequence of motion primitives (2/3)

- The result has the form

$$\Delta x_1(\epsilon) = \sum_{i=1}^{k} r_i \left(\epsilon - 1\right)^{i-1} + O\left(\left|\epsilon - 1\right|^k\right)$$

$$\Delta x_2(\epsilon) = \sum_{i=1}^{k} s_i \left(\epsilon - 1\right)^{i-1} + O\left(\left|\epsilon - 1\right|^k\right)$$

  where $r, s \in \mathbb{R}^k$ are linear in $a, b \in \mathbb{R}^k$

- To achieve $\Delta x_1 = \Delta x_2 = 1$ with error of order k in $|\epsilon - 1|$:

$$r = s = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

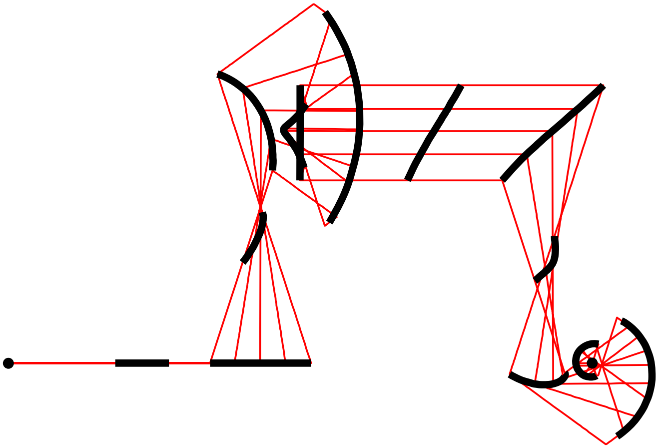
Sequence of motion primitives (3/3)

- Require exactly $k+1$ primitives to achieve kth-order error
- Precompute a, b as $2k$ linear equations in $2k$ variables
- For $\phi = \pi/2$, this can be done in closed form
- By linearity, $a\Delta x_1$ and $b\Delta x_2$ reach arbitrary Δx_1 and Δx_2
- "Planning" means computing piecewise-constant inputs

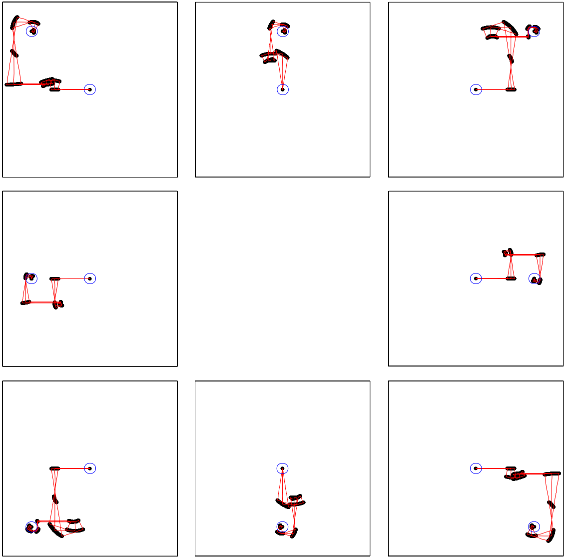
$$a' = \frac{1}{2}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} \Delta x_1 + \begin{bmatrix} 0 \\ b \end{bmatrix} \Delta x_2\right)$$
$$b' = \frac{1}{2}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} \Delta x_1 - \begin{bmatrix} 0 \\ b \end{bmatrix} \Delta x_2\right).$$

- This approach does *not* require sampling ϵ

Example results in simulation



Scaled primitives get you everywhere “for free”



Example results in experiment

Outline

1. Introduction
2. Linear example
3. Nonlinear example
4. Conclusion
   - Connections

Connections

- NMR spectroscopy and imaging (Brockett, Khaneja, Li, Altafini, Beauchard, Coron, Pereira da Silva, Rouchon, etc.)
- Robust control (Dullerud and Paganini, Singer and Seering, Fischer and Psiaki, etc.)
- Sensorless manipulation (Erdmann, Mason, Goldberg, Lynch, Murphey, Akella, van der Stappen, Moll, etc.)
- POMDP planning (Hsu, Hutchinson, Roy, Thrun, etc.)
- Optimal control of micro/nano-robot teams (under review)
- Control-theoretic approach to manipulation of deformable objects (in preparation)
- BMIs based on inverse optimal control

Support

- NSF-CNS-0931871
- NSF-CMMI-0956362-EAGER
- NSF-CMMI-0955088-CAREER

Questions?

Connection to NMR spectroscopy and imaging

- Need to manipulate an ensemble of $\approx 10^{23}$ nuclear spins, each governed by the Schrödinger equation
- Control input changes the potential energy in the system Hamiltonian (e.g., by applying electromagnetic pulses)
- One model is

$$\frac{dx(t,s)}{dt} = \left(A(s) + \sum_i u_i B_i(s) \right) x(t,s)$$

- "s" describes variation in A and B_i from Larmor dispersion (in natural frequencies), rf inhomogeneity (in strength of the applied radio frequency), and relaxation rates
- Is it possible to steer from one state to another?